INVARIANT TRANSFORMATION OF THE EQUATIONS OF MOTION OF AN IDEAL MONATOMIC GAS AND NEW CLASSES OF THEIR EXACT SOLUTIONS

(INVARIANTNOE PREOBRAZOVANIE URAVNENII DVIZHENIIA IDEAL'NOGO ODNOATOMNOGO GAZA I NOVYE KLASSY IKH TOCHNOGO RESHENIIA)

PMM Vol.27, No.3, 1963, pp. 496-508

A.A. NIKOL'SKII
(Moscow)

(Received July 20, 1962)

In this paper a transformation of the equations of motion of an ideal monatomic gas to the coordinates of a uniformly expanding space is investigated. With corresponding transformations of time, velocity fields, pressures, density and temperature, equations of motion are obtained in terms of the new variables which are the same as those in the stationary coordinates. This allows us to extend the entire ordinary gas dynamics into the dynamics of an expanding gas and to juxtapose new solutions with all the available exact solutions of gas dynamics. In the new gas motions considered, the processes taking place over an infinite interval of time t (up to $t = \infty$) are realized, which are similar to those that in the original ordinary gas motions take place in a finite interval of time.

In Section 1, gas motions with surfaces of discontinuity (strong discontinuity, tangential discontinuity) are investigated in a general manner. Basic theorems for such flows are proved.

In Section 2, examples of the new exact solutions are investigated, namely:

a) the uniform expansion and uniform compression of a gas;

b) source or sink type flow;

c) "simple" waves in uniformly expanding and in uniformly compressing gas.

In Sections 3 and 4 the problem of a point explosion in a uniformly expanding and in a uniformly compressing gas is investigated.

1. Formulation of the problem and the basic theorems. For a monatomic gas the adiabetic index γ is equal to 5/3 and, therefore, the general equations of motion of such a gas, if external forces, friction and heat conduction are absent, have the form [1]

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad (uvw,\ xyz) \tag{1.1}$$

$$\frac{\partial \ln \rho}{\partial t} + u\frac{\partial \ln \rho}{\partial x} + v\frac{\partial \ln \rho}{\partial y} + w\frac{\partial \ln \rho}{\partial z} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{1.2}$$

$$\frac{\partial}{\partial t}\frac{p}{\rho^{5/3}} + u\frac{\partial}{\partial x}\frac{p}{\rho^{5/3}} + v\frac{\partial}{\partial y}\frac{p}{\rho^{5/3}} + w\frac{\partial}{\partial z}\frac{p}{\rho^{5/3}} = 0 \tag{1.3}$$

where t is time, p is pressure, ρ is density, x, y and z are the Cartesian coordinates; u, v and w are the vector components of the velocity with respect to the x-, y- and z-axes; symbols (xyz) and (uvw) denote that the remaining equations are obtained by cyclic permutation.

Let us introduce the new independent variables

$$\tau = a - \frac{b^2}{t-c}, \qquad \xi = \frac{b}{t-c}x, \qquad \eta = \frac{b}{t-c}y, \qquad \zeta = \frac{b}{t-c}z \tag{1.4}$$

and the new functions

$$u' = \frac{t-c}{b}u - \frac{x}{b}, \qquad v' = \frac{t-c}{b}v - \frac{y}{b}, \quad w' = \frac{t-c}{b}w - \frac{z}{b} \tag{1.5}$$

$$\rho' = \left(\frac{t-c}{b}\right)^3 \rho, \qquad p' = \left(\frac{t-c}{b}\right)^5 p \tag{1.6}$$

where a, b and c are arbitrary constants with the dimension of time. Also, the following quantities are valid

$$u = \frac{a-\tau}{b}u' + \frac{\xi}{b}, \qquad v = \frac{a-\tau}{b}v' + \frac{\eta}{b}, \qquad w = \frac{a-\tau}{b}w' + \frac{\zeta}{b} \tag{1.7}$$

$$\rho = \left(\frac{a-\tau}{b}\right)^3 \rho', \qquad p = \left(\frac{a-\tau}{b}\right)^5 p', \qquad \frac{b}{t-c} = \frac{a-\tau}{b} \tag{1.8}$$

For the operation of differentiation we have

$$\frac{\partial}{\partial t} = \frac{b^2}{(t-c)^2}\frac{\partial}{\partial \tau} - \frac{b}{(t-c)^2}\left(x\frac{\partial}{\partial \xi} + y\frac{\partial}{\partial \eta} + z\frac{\partial}{\partial \zeta}\right), \qquad \frac{\partial}{\partial x} = \frac{a-\tau}{b}\frac{\partial}{\partial \xi} \tag{1.9}$$

$$\frac{\partial}{\partial y} = \frac{a-\tau}{b}\frac{\partial}{\partial \eta}, \qquad \frac{\partial}{\partial z} = \frac{a-\tau}{b}\frac{\partial}{\partial \zeta}, \qquad \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} = \frac{b^2}{(t-c)^2}\frac{\partial}{\partial \tau} -$$

$$- \frac{b}{(t-c)^2}\left(x\frac{\partial}{\partial \xi} + y\frac{\partial}{\partial \eta} + z\frac{\partial}{\partial \zeta}\right) + \frac{b}{t-c}\left(u\frac{\partial}{\partial \xi} + v\frac{\partial}{\partial \eta} + w\frac{\partial}{\partial \zeta}\right)$$

The last relation may be transformed into the form

$$\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} = \left(\frac{a-\tau}{b}\right)^2\left(\frac{\partial}{\partial \tau} + u'\frac{\partial}{\partial \xi} + v'\frac{\partial}{\partial \eta} + w'\frac{\partial}{\partial \zeta}\right) \quad (1.10)$$

If we substitute expressions (1.7) and (1.8) into equations (1.1) to (1.3) then, by the rules of differentiation (1.9) and (1.10), we obtain equations which are identical to equations (1.1) to (1.3) if in the latter we replace t, x, y, z, p and ρ by τ, ξ, η, ζ, p', ρ'. Therefore, the system of equations (1.1) to (1.3) is invariant with respect to the transformation of variables (1.4) to (1.6). Here the following theorem applies:

Theorem 1.1. If the set of functions

$$u = u'(t, x, y, z), \qquad v = v'(t, x, y, z), \qquad w = w'(t, x, y, z)$$
$$\rho = \rho'(t, x, y, z), \qquad p = p'(t, x, y, z) \quad (1.11)$$

satisfies the system of equations (1.1) to (1.3), then the same system of equations is satisfied by the set of functions

$$u = u_0(t, x, y, z) = \frac{a-\tau}{b}u'(\tau, \xi, \eta, \zeta) + \frac{\xi}{b} \quad (1.12)$$
$$v = v_0(t, x, y, z) = \frac{a-\tau}{b}v'(\tau, \xi, \eta, \zeta) + \frac{\eta}{b}$$
$$w = w_0(t, x, y, z) = \frac{a-\tau}{b}w'(\tau, \xi, \eta, \zeta) + \frac{\zeta}{b}$$
$$\rho = \rho_0(t, x, y, z) = \left(\frac{a-\tau}{b}\right)^3 \rho'(\tau, \xi, \eta, \zeta)$$
$$p = p_0(t; x, y, z) = \left(\frac{a-\tau}{b}\right)^5 p'(\tau, \xi, \eta, \zeta)$$
$$\tau = a - \frac{b^2}{t-c}, \qquad \xi = \frac{b}{t-c}x, \qquad \eta = \frac{b}{t-c}y, \qquad \zeta = \frac{b}{t-c}z \quad (1.13)$$
$$\sqrt{\xi^2 + \eta^2 + \zeta^2} = \frac{b}{t-c}\sqrt{x^2 + y^2 + z^2} \quad (1.14)$$

where, as above, a, b and c are arbitrary constants. Here, from the condition that ρ_0 and p_0 are positive, it follows that the new solution may be considered only in the region where $(a - \tau)/b > 0$, i.e. $b/(t-c) > 0$.

The validity of Theorem 1.1 may also be verified by direct substitution of expressions (1.12) to (1.14) into the system (1.1) to (1.3), using the condition that the functions (1.11) satisfy this system.

Let E' denote the motion determined by the functions (1.11)

$$\Omega' = \Omega'(t, x, y, z) = \{u'(t, x, y, z), v'(t, x, y, z), w'(t, x, y, z)\}$$
$$\rho'(t, x, y, z), \qquad p'(t, x, y, z) \quad (1.15)$$

and E_0 the motion which is given by the functions (1.12), so that

$$\Omega_0 = \Omega_0(t, x, y, z) = \{u_0(t, x, y, z), v_0(t, x, y, z)\, w_0(t, x, y, z)\}$$
$$\rho_0(t, x, y, z), \qquad p_0(t, x, y, z) \tag{1.16}$$

where Ω' and Ω_0 are the velocity vectors of the motions. Let us investigate now a gas motion with the surfaces Σ of a strong discontinuity

$$F(t, x, y, z) = 0 \tag{1.17}$$

The surface Σ separates the space into two regions: on one side of this surface $F(x, y, z, t) < 0$ and on the other side $F(t, x, y, z) > 0$.

As in [1], we shall call the first region negative; let the values to which any scalar or vector function $b(t, x, y, z)$ tends, when it approaches Σ from the side of the negative region, be denoted by b_-; in the other region, called the positive region, the corresponding values of b on Σ shall be denoted by b_+. Let us also introduce the notation

$$b_+ - b_- = [b]$$

At the surfaces Σ of the strong discontinuity with equation (1.17), in the case of motion E', the conditions of dynamic consistency which are satisfied ([1], Chap. 1, Section 2) are

$$\left[\rho'\left(\frac{\partial F}{\partial t} + u'\frac{\partial F}{\partial x} + v'\frac{\partial F}{\partial y} + w'\frac{\partial F}{\partial z}\right)\right] = 0 \tag{1.18}$$

$$\rho'\left(\frac{\partial F}{\partial t} + u'\frac{\partial F}{\partial x} + v'\frac{\partial F}{\partial y} + w'\frac{\partial F}{\partial z}\right)[\Omega'] = -[p']\left(\frac{\partial F}{\partial x}\mathbf{i} + \frac{\partial F}{\partial y}\mathbf{j} + \frac{\partial F}{\partial z}\mathbf{k}\right) \tag{1.19}$$

$$\rho'\left(\frac{\partial F}{\partial t} + u'\frac{\partial F}{\partial x} + v'\frac{\partial F}{\partial y} + w'\frac{\partial F}{\partial z}\right)\left[\frac{\Omega'\Omega'}{2} + \frac{5}{2}\frac{p'}{\rho'}\right] =$$
$$= -\left[p'\left(u'\frac{\partial F}{\partial x} + v'\frac{\partial F}{\partial y} + w'\frac{\partial F}{\partial z}\right)\right] \tag{1.20}$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors along the axes x, y and z. In addition, the theorem of Tsemplin ([1], Chap. 1, Section 5) must be satisfied

$$\left[u'\frac{\partial F}{\partial x} + v'\frac{\partial F}{\partial y} + w'\frac{\partial F}{\partial z}\right] < 0 \tag{1.21}$$

Theorem 1.2. In the case of motions E_0 the surface of a strong discontinuity (where the conditions of dynamic consistency are satisfied) will be the surface Σ_0 with the equations

$$F(\tau, \xi, \eta, \zeta) = 0$$
$$\tau = a - \frac{b^2}{t-c}, \qquad \xi = \frac{b}{t-c}x, \qquad \eta = \frac{b}{t-c}y, \qquad \zeta = \frac{b}{t-c}z \tag{1.22}$$

To prove this theorem we shall obtain first an auxiliary relation; performing a scalar multiplication of (1.19) by $\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$, we have

$$\rho'\left(\frac{\partial F}{\partial t} + u'\frac{\partial F}{\partial x} + v'\frac{\partial F}{\partial y} + w'\frac{\partial F}{\partial z}\right)[\boldsymbol{\Omega}\times\mathbf{r}] = -[p']\left(\frac{\partial F}{\partial x}x + \frac{\partial F}{\partial y}y + \frac{\partial F}{\partial z}z\right) \quad (1.23)$$

Let us make the following substitutions in the relations (1.18) to (1.20) and (1.21): in place of function $F(t, x, y, z)$ we substitute function $F_0(t, x, y, z) = F(\tau, \xi, \eta, \zeta)$, and instead of u', v', w', ρ', p' we substitute functions u_0, v_0, w_0, ρ_0, p_0, determined by the relations (1.12), (1.13) and (1.14). We also substitute for the derivatives with respect to t, x, y and z the derivatives with respect to τ, ξ, η, ζ, using the differentiation rules (1.9) and (1.10). Then, using equations (1.20) and (1.23) and also the relation $[\mathbf{r} \times \mathbf{r}] = 0$ in the transformations, we obtain equations which are obtained from equations (1.18) to (1.21) by the formal replacement of t, x, y, z, u', v', w' by τ, ξ, η, ζ, u_0, v_0, w_0, respectively.

In this way, conditions (1.18) to (1.21) which are valid on the surfaces $F(t, x, y, z) = 0$ in the case of motion E', will be equivalent to the conditions described on the surfaces $F(\tau, \xi, \eta, \zeta) = 0$ in the case of motion E_0.

Hence it follows that the dynamic conditions of consistency are satisfied on the surfaces Σ_0 for the motion E_0, provided they are satisfied on the surfaces Σ' for the motion E'. Thus, the Theorem 1.2 is proved. If the surface Σ' with equation (1.17) is the surface of a stationary (tangential) discontinuity for the motion E', then the surface Σ_0 with equation (1.22) will also be the surface of a tangential discontinuity; indeed, if on the surface of the motion E' the conditions are satisfied that the Σ' surface at all times consists of the same particles in each of the regions separated by it

$$\frac{\partial F}{\partial t} + u_+\frac{\partial F}{\partial x} + v_+\frac{\partial F}{\partial y} + w_+\frac{\partial F}{\partial z} = 0, \quad \frac{\partial F}{\partial t} + u_-\frac{\partial F}{\partial x} + v_-\frac{\partial F}{\partial y} + w_-\frac{\partial F}{\partial z} = 0 \quad (1.24)$$

and the conditions of dynamic consistency

$$[p] = 0 \quad (1.25)$$

then they are satisfied on the surface Σ_0 of the motion E_0; this may easily be verified by the corresponding substitutions.

If the surface $F(t, x, y, z) = 0$ of the motion E' is a non-penetrated boundary, then the surface $F(\tau, \xi, \eta, \zeta) = 0$ will be a non-penetrated boundary in the motion E_0.

The results obtained allow the juxtaposition of any motions E' of an ideal gas, with and without strong discontinuities, with motions E_0.

2. Examples of new exact solutions. a) Let $u' = v' = w' = 0$, $\rho' = \rho'(x, y, z)$, $p' = p_1 = \text{const}$, that is, E' corresponds to a state of rest. For E_0 we obtain

$$u_0 = \frac{x}{t-c}, \qquad v_0 = \frac{y}{t-c}, \qquad w_0 = \frac{z}{t-c} \tag{2.1}$$

$$\rho_0 = \left(\frac{b}{t-c}\right)^3 \rho' \left(\frac{b}{t-c}x, \frac{b}{t-c}y, \frac{b}{t-c}z\right), \qquad p_0 = \left(\frac{b}{t-c}\right)^5 p_1 \tag{2.2}$$

where ρ' is an arbitrary positive function of the arguments. These motions were obtained by Sedov [2] for arbitrary γ and $p' = \text{const}$, as a special case of similarity gas motions having central symmetry. However, these motions deserve particular attention because they possess a property of homogeneity: the velocity fields of the particles relative to a coordinate system fixed with any moving material particle are the same, and thus the point $x = y = z = 0$ in this sense does not differ from any other point. Indeed, for any fixed particle x_*, y_*, z_*, equations (2.1) obviously give the following law of motion

$$\frac{dx_*}{dt} = \frac{x_*}{t-c}, \qquad \frac{dy_*}{dt} = \frac{y_*}{t-c}, \qquad \frac{dz_*}{dt} = \frac{z_*}{t-c} \tag{2.3}$$

$$\frac{x_*}{t-c} = u_* = \text{const}, \qquad \frac{y_*}{t-c} = v_* = \text{const}, \qquad \frac{z_*}{t-c} = w_* = \text{const} \tag{2.4}$$

Any particle under consideration moves with a constant velocity u_*, v_*, w_*. Every particle moves at constant velocity, but the velocities of two different particles are different. Fixing the system of coordinates to a given particle, let

$$\begin{aligned} x_1 &= x - x_*, \qquad & y_1 &= y - y_*, \qquad & z_1 &= z - z_* \\ u_1 &= u - u_*, \qquad & v_1 &= v - v_*, \qquad & w_1 &= w - w_* \end{aligned}$$

Then equations (2.1) will take the form

$$u_1 = \frac{x_1}{t-c}, \qquad v_1 = \frac{y_1}{t-c}, \qquad w_1 = \frac{z_1}{t-c}$$

which is what was asserted. Equations (2.1) and (2.2) for $c < t$ and $b > 0$ give a uniform expansion of a gas, and for $c > t$ and $b < 0$ they give a uniform compression of a gas.

In the case of motions in uniform expansions and compressions, characterized by equations (2.1) and (2.2), the velocity vector at every moment of time in all space is $\Omega = (t - c)^{-1}\mathbf{r}$ (proportional to the radius vector $\mathbf{r}$), and the pressure p is a function of time only.

If it is known that a certain motion at some moment of time $t = t_0$

possesses these properties, then it will possess them also at $t > t_0$.

Let us solve the following Cauchy problem. Let

$$u = \lambda x, \qquad v = \lambda y, \qquad w = \lambda z \qquad (\lambda = \text{const})$$
$$\rho = f(x, y, z), \qquad p = p_0 = \text{const} \qquad \text{for } t = t_0$$

We shall find the motion for $t > t_0$. Putting

$$b = 1 / \lambda, \qquad c = t_0 - 1 / \lambda$$

in equations (2.1) and (2.2), and replacing the arbitrary function ρ' by f, we obtain the solution of the given Cauchy problem

$$u = \frac{\lambda x}{\lambda (t - t_0) + 1}, \qquad v = \frac{\lambda y}{\lambda (t - t_0) + 1}, \qquad w = \frac{\lambda z}{\lambda (t - t_0) + 1} \tag{2.5}$$

$$\rho = \frac{1}{(\lambda t - \lambda t_0 + 1)^3} f\left(\frac{x}{\lambda t - \lambda t_0 + t_0}, \ \frac{y}{(\lambda t - \lambda t_0 + 1)}, \ \frac{z}{(\lambda t - \lambda t_0 + 1}\right) \tag{2.6}$$

$$p = \frac{1}{(\lambda t - \lambda t_0 + 1)^5} p_0 \tag{2.7}$$

Thus, the motion for $t > t_0$ is indeed the motion of uniform expansion or compression.

b) Motion E' will be a steady spherical source or sink flow of an ideal monatomic gas. It is described by the well-known relations [1]

$$4\pi r^2 \rho v = Q, \qquad \rho = \rho_0 \left(1 - \frac{1}{10} \frac{\rho_0}{p_0} v^2\right)^{3/2}, \qquad p = p_0 \left(\frac{\rho}{\rho_0}\right)^{5/3}$$
$$Q = \text{const}, \qquad \rho_0 = \text{const}, \qquad p_0 = \text{const} \tag{2.8}$$

According to Theorem 1.1, in order to obtain motion E_0, we have to replace v, ρ, p and r by

$$\frac{t - c}{b}\left(v - \frac{r}{t - c}\right), \qquad \left(\frac{t - c}{b}\right)^3 \rho, \qquad \left(\frac{t - c}{b}\right)^5 p, \qquad \frac{b}{t - c} r$$

respectively.

The new solution will be non-steady and it is defined by the relation

$$4\pi \left(\frac{t - c}{b}\right)^2 r^2 \rho \left(v - \frac{r}{t - c}\right) = Q$$
$$\rho = \rho_0 \left(\frac{b}{t - c}\right)^3 \left[1 - \frac{1}{10} \frac{\rho_0}{p_0} \left(\frac{t - c}{b} v - \frac{r}{b}\right)^2\right]^{3/2}, \qquad p = p_0 \left(\frac{\rho}{\rho_0}\right)^{5/3} \tag{2.9}$$

c) Motion E' is a "simple" Riemann wave; the relation between quantities for waves of the two types is given by the relations [1]

$$x = \left(\tfrac{4}{3} u + \alpha\right) t + f(u), \qquad v = 0, \qquad w = 0 \qquad (\alpha = \text{const}) \tag{2.10}$$

$$\beta p^{1/5} = \pm \frac{1}{3} u \pm \alpha, \qquad \rho = \frac{5}{3} \frac{1}{\beta^2} p^{3/5} \qquad (\beta = \text{const}) \tag{2.11}$$

where $f(u)$ is an arbitrary function of the arguments. For the motion E_0, using Theorem 1.1 and choosing arbitrary constants of the transformation of variables, we obtain in connection with case (a) "simple" waves in a uniformly expanding and compressing gas

$$\frac{x}{\lambda(t-t_0)+1} = \left\{\tfrac{4}{3}\left[(\lambda t - \lambda t_0 + 1)u - \lambda x\right] + \alpha\right\}\tau + f\left[(\lambda t - \lambda t_0 + 1)u - \lambda x\right] \tag{2.12}$$

$$v = \frac{\lambda y}{\lambda(t-t_0)+1}, \qquad w = \frac{\lambda z}{\lambda(t-t_0)+1}, \qquad \rho = \frac{5}{3}\frac{1}{\beta^2}p^{3/5}, \qquad \tau = \frac{t-t_0}{\lambda(t-t_0)+1}$$

$$\beta(\lambda t - \lambda t_0 + 1)p^{1/5} = \pm\tfrac{1}{3}\left[(\lambda t - \lambda t_0 + 1)u - \lambda x\right] \pm \alpha$$

Taking the plus sign on the right-hand side of the first of equations (2.11) and assuming $f(u) \equiv 0$, $\alpha = \beta p_1^{1/5}$, $p_1 = \text{const}$, we obtain the known solution of the problem of motion of a gas for $t > 0$, where $t = 0$

for the half-space $x > 0$

$$u = 0, \qquad p = p_1 = \text{const}, \qquad \rho = \rho_1 = \frac{5}{3}\frac{1}{\beta^2}p_1^{3/5}$$

for the half-space $x < 0$ $\quad p = 0 \quad \rho = 0$

$$u = \frac{3}{4}\left[\frac{x}{t} - \left(\frac{5}{3}\frac{p_1}{\rho_1}\right)^{1/2}\right], \qquad v = w = 0, \tag{2.13}$$

$$\frac{x}{t} < \left(\frac{5}{3}\frac{p_1}{\rho_1}\right)^{1/2}, \quad \left(\frac{5}{3}\frac{p_1}{\rho_1}\right)^{1/2}\left(\frac{p}{p_1}\right)^{1/5} = \frac{1}{4}\frac{x}{t} + \frac{3}{4}\left(\frac{5}{3}\frac{p_1}{\rho_1}\right)^{1/2}$$

The corresponding motion E_0 is given by the equations

$$u = \frac{3}{4}\left[\frac{x}{t-t_0} - \left(\frac{5}{3}\frac{p_1}{\rho_1}\right)^{1/2}\right](\lambda t - \lambda t_0 + 1)^{-1} + \frac{\lambda x}{\lambda(t-t_0)+1}$$

$$v = \frac{\lambda y}{\lambda(t-t_0)+1}, \qquad w = \frac{\lambda z}{\lambda(t-t_0)+1} \tag{2.14}$$

$$\left(\frac{5}{3}\frac{p_1}{\rho_1}\right)^{1/2}\left(\frac{p}{p_1}\right)^{1/5} = \frac{1}{\lambda(t-t_0)+1}\left[\frac{1}{4}\frac{x}{t-t_0} + \frac{3}{4}\left(\frac{5}{3}\frac{p_1}{\rho_1}\right)^{1/2}\right]$$

These relations give the solution of the problem of the gas motion for $t > t_0$, which at the time $t = t_0$ in the half-space $x < 0$ satisfied the conditions $p = 0$, $\rho = 0$, and in the half-space $x > 0$ satisfied the relations $u = \lambda x$, $v = \lambda y$, $w = \lambda z$, $\rho = \rho_1 = \text{const}$, $p = p_1 = \text{const}$.

It is interesting that the velocity of the gas particles next to the boundary of the region of zero pressure is the same for the motions E' and E_0 being equal to $3(5/3\ p_1/\rho_1)^{1/2}$.

3. Point explosion in a uniformly expanding and in a uniformly compressing gas. In the motion E', for $t = 0$

$$u = v = w = 0, \qquad \rho = \rho_1 = \text{const}, \qquad p = p_1 = \text{const}$$

At the time $t = 0$ at the point $x = y = z = 0$, energy E is instantaneously liberated [2]. For time $t > 0$ there are two regions, separated by a shock wave: a region of disturbed motion next to the point $r = 0$ $[r = \sqrt{(x^2 + y^2 + z^2)}]$, and a region of undisturbed motion. We shall write the law of the variation of the distance r_2 of the wave from the point $r = 0$ in the form

$$r_2 = \left(\frac{E}{p_1}\right)^{1/3} F(t''), \qquad t'' = E^{-1/3}\rho_1^{-1/2}p_1^{5/6}\,t \qquad (F(0) = 0) \qquad (3.1)$$

Considering motion with spherical symmetry, let v denote the projection of the velocity vector on the radius vector. For the velocity v_+, density ρ_+ and pressure p_+, the following relations [2] hold behind the wave

$$v_+ = \frac{3}{4}\left(\frac{p_1}{\rho_1}\right)^{1/2}\left\{N(t'') - \frac{5}{3}\frac{1}{N(t'')}\right\}, \qquad N(t'') = \frac{dF(t'')}{dt''}$$

$$N(\infty) = \sqrt{\frac{5}{3}}, \qquad N(0) = \infty \qquad (3.2)$$

$$\frac{\rho_+}{\rho_1} = 4\left\{1 + \frac{5}{N(t'')^2}\right\}^{-1}, \qquad \frac{p_+}{p_1} = \frac{3}{4}\left\{N(t'')^2 - \frac{1}{3}\right\} \qquad (3.3)$$

We shall assume that the motion inside the disturbed region is given in the form

$$v = v_+ V(t'', r/r_2), \qquad \rho = \rho_+ R(t'', r/r_2) \qquad p = p_+ P(t'', r/r_2)$$

$$V(t'', 1) = 1, \qquad R(t'', 1) = 1, \quad P(t'', 1) = 1 \qquad (3.4)$$

At time $t'' \ll 1$ the main terms for $F(t'')$ and $N(t'')$ will be the expressions

$$F(t'') = \alpha^{-1/5}t''^{2/5}, \qquad N(t'') = \frac{2}{5}\alpha^{-1/5}t''^{-3/5} \qquad (\alpha = 0.487) \qquad (3.5)$$

where the given value of α was obtained by numerical calculation [3].

The main terms in the expressions (3.1) to (3.3) will be

$$r_2 = \left(\frac{E}{\alpha\rho_1}\right)^{1/5}t^{2/5}, \quad v_+ = \frac{3}{10}\left(\frac{E}{\alpha\rho_1}\right)^{1/5}t^{-3/5} \quad \rho_+ = 4\rho_1, \quad p_+ = \frac{3}{25}\rho_1\left(\frac{E}{\alpha\rho_1}\right)^{2/5}t^{-6/5} \qquad (3.6)$$

The last expression will be exact if we put $p_1 = 0$ (strong explosion). Inside the region of disturbance we have the expressions [3]

$$\frac{v}{v_+} = \frac{r}{r_2}\mu, \qquad \frac{r}{r_2} = \frac{(5\mu - 4)^{2/13}}{\mu^{2/5}}\left(\frac{5}{2} - \frac{3}{2}\mu\right)^{-82/195}, \qquad 4/ \leqslant \mu \leqslant 1) \qquad (3.7)$$

$$\frac{\rho}{\rho_+} = \frac{(5\mu - 4)^{9/13}}{(4 - 3\mu)^6}\left(\frac{5}{2} - \frac{3}{2}\mu\right)^{82/13}, \qquad \frac{p}{p_+} = \frac{\mu^{6/5}}{(4 - 3\mu)^5}\left(\frac{5}{2} - \frac{3}{2}\mu\right)^{82/15}$$

Using Theorem 1.2, equations (1.22) and equation (3.1), we find that in the case of motion E_0 a shock wave is propagating in the gas according to the law

$$r_2 = \frac{t-c}{b}\left(\frac{E}{p_1}\right)^{1/3} F(\tau''), \qquad \tau'' = E^{-1/3}\rho_1^{-1/2}p_1^{5/6}\left(a - \frac{b^2}{t-c}\right) \tag{3.8}$$

$$a = \text{const}, \quad b = \text{const}, \quad c = \text{const}$$

Outside the spherical shock wave (in front of it) we obtain from equations (1.12), (1.13) and (1.14)

$$v = \frac{r}{t-c}, \qquad \rho = \rho_1\left(\frac{b}{t-c}\right)^3, \qquad p = p_1\left(\frac{b}{t-c}\right)^5 \tag{3.9}$$

where v is the projection of the velocity vector on the radius vector. This indicates that a uniform expansion or compression of the gas takes place. We shall select the constants a, b and c such that motion E_0 will give a solution of the problem of the gas motion according to the initial conditions

$$u = \lambda x, \quad v = \lambda y, \quad w = \lambda z \quad (\lambda = \text{const}); \qquad \rho = \rho_1, \quad p = p_1 \neq 0 \text{ for } t = t_0$$

The energy E is instantaneously liberated at the point $r = 0$ at $t = t_0$. In view of equations (2.5), (2.6) and (2.7) and their derivation, put $b = 1/\lambda$, $c = t_0 - 1/\lambda$. Requiring that, in equation (3.8), $r_2 = 0$ for $t = t_0$, we also put $a = 1/\lambda$. Then equations (3.8) and (3.9) will have the form

$$r_2 = (\lambda t - \lambda t_0 + 1)\left(\frac{E}{p_1}\right)^{1/3} F(\tau''), \qquad \tau'' = E^{-1/3}\rho_1^{-1/2}p_1^{5/6}\frac{t-t_0}{\lambda(t-t_0)+1} \tag{3.10}$$

$$v = \frac{\lambda r}{\lambda(t-t_0)+1}, \qquad \rho = \frac{\rho_1}{(\lambda t - \lambda t_0 + 1)^3}, \qquad p = \frac{p_1}{(\lambda t - \lambda t_0 + 1)^5} \tag{3.11}$$

For small values of τ'' (values of t near t_0) equation (3.10) is identical to equation (3.1) in its main term, if in the latter equation we replace t by $t - t_0$. From this it follows that in equations (3.10) and (3.11) the quantity E is equal to the energy which is supplied instantaneously at $t = t_0$ in the motion E_0, i.e. in the uniformly expanding or compressing gas. Using equations (1.12), (1.13), (1.14) and (3.10) for the motion E_0, instead of equations (3.2) to (3.4), we obtain

$$v = \frac{3}{4}\left(\frac{p_1}{\rho_1}\right)^{1/2}\left\{N(\tau'') - \frac{5}{3}\frac{1}{N(\tau'')}\right\}\frac{V(\tau'', r/r_2)}{\lambda(t-t_0)+1} + \frac{\lambda r}{\lambda(t-t_0)+1} \tag{3.12}$$

$$\rho = 4\frac{\rho_1 R(\tau'', r/r_2)}{[\lambda(t-t_0)+1]^3}\left\{1 + \frac{5}{N(\tau'')^2}\right\}^{-1}, \quad p = \frac{3}{4}\frac{p_1 P(\tau'', r/r_2)}{[\lambda(t-t_0)+1]^5}\left\{N(\tau'')^2 - \frac{1}{3}\right\} \tag{3.13}$$

$$v_+ = \frac{3}{4}\left(\frac{p_1}{\rho_1}\right)^{1/2}\left\{N(\tau'') - \frac{5}{3}\frac{1}{N(\tau'')}\right\}(\lambda t - \lambda t_0 + 1)^{-1} + \lambda\left(\frac{E}{p_1}\right)^{1/3} F(\tau'') \quad (3.14)$$

$$\frac{\rho_+}{\rho_1(\lambda t - \lambda t_0 + 1)^{-3}} = 4\left\{1 + \frac{5}{N(\tau'')^2}\right\}^{-1}, \qquad \frac{p_+}{p_1(\lambda t - \lambda t_0 + 1)^{-5}} = \frac{3}{4}\left\{N(\tau'')^2 - \frac{1}{3}\right\} \quad (3.15)$$

Let us first investigate the case $\lambda > 0$ (explosion in an expanding gas). From the second of equations (3.10) we have for $\lambda > 0$

$$\tau_\infty'' = \lim_{t\to\infty} \tau'' = \frac{1}{\lambda} E^{-1/3}\rho_1^{-1/2}p_1^{5/6} \quad (3.16)$$

From the equations (3.10) and (3.11) we find that the mass of the gas M, confined inside the sphere which coincides with the shock wave, is equal to

$$M = \frac{4}{3}\pi\rho_1 \frac{E}{p_1} F(\tau'')^3 \quad (3.17)$$

This mass remains bounded when t increases

$$\lim_{t\to\infty} M = M_\infty = \frac{4}{3}\pi\rho_1 \frac{E}{p_1} F(\tau_\infty'')^3 \quad (3.18)$$

Equation (3.14) leads to

$$\lim_{t\to\infty} v_+ = v_{+\infty} = \lambda\left(\frac{E}{p_1}\right)^{1/3} F(\tau_\infty'') \quad (3.19)$$

Differentiating equation (3.10) and going over to the limit we obtain

$$\lim \frac{dr_2}{dt} = N_\infty = \lambda\left(\frac{E}{p_1}\right)^{1/3} F(\tau_\infty'') \quad (3.20)$$

Let us designate the velocity of a gas immediately after the shock wave by v_-. In the first of equations (3.11) we assume $r = r_2$ (where r_2 is given by the equation (3.10)) and go over to the limit; we obtain

$$\lim_{t\to\infty} v_- = v_{-\infty} = \lambda\left(\frac{E}{p_1}\right)^{1/3} F(\tau_\infty'') \quad (3.21)$$

Equations (3.19), (3.20) and (3.21) show that at $t \to \infty$ the velocity of shock wave propagation relative to the moving particles of the gas undisturbed by the shock wave approaches zero. Also the increase of particle velocity due to the shock passage approaches zero. The third of equations (3.11) shows that the denominator on the left side of the second equation of (3.15) represents the pressure p_- in front of the shock wave; we have

$$\lim_{t\to\infty} \frac{p_+}{p_-} = \frac{3}{4}\left\{N(\tau_\infty'')^2 - \frac{1}{3}\right\} \quad (3.22)$$

When $t \to \infty$ the magnitude of p_- and p_+ approach zero. The right side of equation (3.22) equals 1 only for $\tau_\infty'' = \infty$, for finite values of τ_∞'' it is greater than 1. Therefore, for any combination of the resulting quantities in expression (3.16) for τ_∞'', the ratio p_+/p_- when $t \to \infty$ approaches a finite value greater than unity.

Consider the expression for the kinetic energy E_v of the mass of gas inside the sphere of the shock wave

$$E_v = 4\pi \int_0^{r_2} \rho \frac{v^2}{2} r^2 \, dr = 4\pi r_2^3 \int_0^1 \rho \frac{v^2}{2} \left(\frac{r}{r_2}\right)^2 d\left(\frac{r}{r_2}\right) \tag{3.23}$$

Let us find the limit of E_v for $t \to \infty$. In this limit the influence of the first term on the right side of the equation (3.12) vanishes. Using equations (3.10), (3.12), (3.13) and (3.16), we obtain (3.24)

$$\lim_{t\to\infty} E_v = 8\pi\rho_1\lambda^2 \left(\frac{E}{p_1}\right)^{5/3} F(\tau_\infty'')^5 \left\{1 + \frac{5}{N(\tau''_\infty)^2}\right\}^{-1} \int_0^1 R\left(\tau_\infty'', \frac{r}{r_2}\right)\left(\frac{r}{r_2}\right)^4 d\left(\frac{r}{r_2}\right)$$

The internal energy E_p of the mass of gas inside the sphere of the shock wave, in the case of a monatomic gas ($\gamma = 5/3$), equals

$$E_p = 4\pi \int_0^{r_2} \frac{3}{2} p r^2 dr = 6\pi r_2^3 \int_0^1 p \left(\frac{r}{r_2}\right)^2 d\left(\frac{r}{r_2}\right) \tag{3.25}$$

Using (3.10) and (3.13), we obtain $\lim E_p = 0$ for $t \to \infty$.

If no explosion occurred in the gas of mass M given by the equation (3.17), then the condition of the gas would be determined by equations (3.11). For $t \to \infty$ the internal energy would approach zero, and the kinetic energy obviously would approach the expression

$$\lim_{t\to\infty} E_v = \frac{2}{5} \pi\rho_1\lambda^2 \left(\frac{E}{p_1}\right)^{5/3} F(\tau_\infty'')^5 \tag{3.26}$$

The difference between the right-hand sides of equations (3.24) and (3.26) is equal to the energy of the explosion. Considering this difference and substituting the quantity λ from equation (3.16), we obtain

$$E = 2\pi \frac{E}{\tau_\infty''^2} F(\tau_\infty'')^5 \left\{\left[\frac{1}{4} + \frac{5/4}{N(\tau_\infty'')^2}\right]^{-1} \int_0^1 R\left(\tau_\infty'', \frac{r}{r_2}\right)\left(\frac{r}{r_2}\right)^4 d\left(\frac{r}{r_2}\right) - \frac{1}{5}\right\}$$

or after cancelling the quantity E

$$2\pi \frac{F(\tau_\infty'')^5}{\tau_\infty''^2} \left\{\left[\frac{1}{4} + \frac{5/4}{N(\tau_\infty'')^2}\right]^{-1} \int_0^1 R\left(\tau_\infty', \frac{r}{r_2}\right)\left(\frac{r}{r_2}\right)^4 d\left(\frac{r}{r_2}\right) - \frac{1}{5}\right\} = 1 \tag{3.27}$$

The quantity τ_∞'' may assume, as is seen from the relation (3.16), any values from 0 to ∞. By using this unusual method, we have obtained a certain integral relation for a point explosion in a stagnant monatomic gas, because the functions F, N and R contained in it and defined by the equations (3.1), (3.2) and (3.4) pertain precisely to this case.

The foregoing analysis shows that, in solving the problem of the point explosion in a uniformly expanding gas, and by means of the same universal solution of the problem of the point explosion in a stagnant gas, a one-parameter family of solutions is obtained in which the quantity τ_∞'' appears as the parameter. In solving the problem of a point explosion in an expanding gas, for every fixed value of τ_∞'' up to $t \to \infty$ use is made of only a "part" of the solution of the problem of a point explosion in a stagnant gas, which is determined by the range of variation of the non-dimensional time $0 < t'' < \tau_\infty''$. In the expanding gas there occurs a peculiar "freezing" of the processes, corresponding to the analogous processes taking place in a stagnant gas. This "freezing" may come about at the stage of strong blast if, for the corresponding τ_∞'' in equation (3.3), we have

$$\frac{3}{4}\left\{N\,(\tau_\infty'') - \frac{5}{3}\right\} \gg 1$$

at the stage of "strong blast" with back pressure, if

$$\frac{3}{4}\left\{N\,(\tau_\infty'') - \frac{5}{3}\right\} \approx 1$$

and at the stage of the dying-out of the explosion, if

$$\frac{3}{4}\left\{N\,(\tau_\infty'') - \frac{5}{3}\right\} \ll 1$$

If $\lambda < 0$ (explosion in a uniformly compressing space), then for $t \to t_0 - 1/\lambda = t^\circ$, according to the formula (3.10) $\tau'' \to \infty$, and consequently, in the interval of time $t_0 < t < t_0 - 1/\lambda$, regimes are established similar to those which are realized for an explosion in a stagnant gas in an infinite interval of non-dimensional time: $0 < t'' < \infty$. The second equation of (3.15) gives

$$\lim_{t \to t^\circ} \frac{p_+}{p_-} = \lim_{\tau'' \to \infty} \frac{3}{4}\left\{N\,(\tau'')^2 - \frac{1}{3}\right\} = 1 \qquad (3.28)$$

Therefore, the relative pressure increase in a shock wave for $t \to t_0 - 1/\lambda$ becomes infinitely small. For t'' large the function $F(t'') \to \infty$ as $t''\sqrt{(5/3)}$ (acoustic shock wave). Using this asymptotic value we obtain from equations (3.10)

$$\lim_{t \to t^\circ} r_2 = r_{2\infty} = -\frac{1}{\lambda}\left(\frac{5}{3}\,\frac{p_1}{\rho_1}\right)^{1/2} = -\frac{1}{\lambda}\,a_1 \qquad (3.29)$$

where a_1 is the sound velocity in the compressing gas at $t = t_0$. The essence of relation (3.29) is easily explained. Relations (3.11) characterize the undisturbed state of the compressing gas. In this condition, the region in which the velocity is greater or equal to the sound velocity is described by the expressions

$$r \geqslant -\frac{1}{\lambda}\left(\frac{5}{3}\frac{p_1}{\rho_1}\right)^{1/2} \tag{3.30}$$

The shock wave is propagating through the gas particles, the velocity of which is directed towards the point $r = 0$. If the wave becomes acoustic, its position in space becomes fixed at the value r_2, determined by equation (3.29), where the velocity of the wave propagating through the gas particles is equal to the velocity of motion of the particles toward the point $r = 0$.

It is evident that, for $t \to t_0 - 1/\lambda$, every gas particle enters the sphere of the shock wave, because for $t \to t_0 - 1/\lambda$ any fixed particle outside the shock wave moving according to equation (3.11) approaches arbitrarily close to the point $r = 0$. Expression (3.29) for $r_{2\infty}$ does not depend on the explosion energy E. Also, from physical considerations it is clear that if we fix the quantities λ, ρ_1, p_1, and some instant of time $t = t_1$, $t_0 < t_1 < t_0 - 1/\lambda$, then, provided E is sufficiently large at the moment $t = t_1$, the quantity r_2 will be arbitrarily large and, in any case, it may be made greater than the quantity $r_{2\infty}$. But this means that for such values of E the radius r_2 of the shock wave sphere for $t > t_0$ first increases in time and then decreases to the value $r_{2\infty}$.

For the motion determined by equations (3.11) (for $\lambda < 0$), let us find the boundary $r = r'(t)$ of the variation in time of the region of infinitely small disturbances, produced at the point $r = 0$ when $t = t_2$, where $t_0 < t_2 < t_0 - 1/\lambda$. We have

$$\frac{dr'}{dt} = \frac{\lambda r'}{\lambda(t - t_0) + 1} + \left(\frac{5}{3}\frac{p_1}{\rho_1}\right)^{1/2}\frac{1}{\lambda(t - t_0) + 1} \tag{3.31}$$

Integrating and requiring that the condition $r' = 0$ at $t = t_2$ be satisfied, we obtain

$$r' = -\frac{1}{\lambda}\left(\frac{5}{3}\frac{p_1}{\rho_1}\right)^{1/2}\frac{t - t_2}{t_0 - 1/\lambda - t_2} \tag{3.32}$$

For $t = t_0 - 1/\lambda$, $r' = r_{2\infty}$, where $r_{2\infty}$ is determined by equation (3.29). Thus the front of the shock wave moves with constant velocity equal to the velocity of sound at $t = t_2$. At $t = t_0 - 1/\lambda$ the sound waves produced at the point $r = 0$ at any arbitrary time reach the sphere $r = r_{2\infty}$. All the acoustic disturbances produced at the point $r = 0$ are concentrated inside this sphere.

4. Exact solution of the problem of a point explosion without back pressure in a uniformly expanding and a uniformly compressing gas. Consider now the case $p_1 = 0$ and $\rho_1 \neq 0$ (strong explosion without back pressure). For convenience we substitute expressions (3.6) into equations (3.7), so that the quantities are expressed in terms of t, r and μ, and obtain

$$\begin{aligned} r &= \left(\frac{E}{\alpha\rho_1}\right)^{1/5} t^{2/5}\mu^{-2/5}(5\mu - 4)^{2/13}\left(\frac{5}{2}-\frac{3}{2}\mu\right)^{-82/195} \\ v &= \frac{3}{10}\left(\frac{E}{\alpha\rho_1}\right)^{1/5} t^{-3/5}\mu^{3/5}(5\mu - 4)^{2/13}\left(\frac{5}{2}-\frac{3}{2}\mu\right)^{-82/195} \\ \rho &= 4\rho_1(5\mu - 4)^{9/13}(4-3\mu)^{-6}\left(\frac{5}{2}-\frac{3}{2}\mu\right)^{82/13} \\ p &= \frac{3}{25}\rho_1\left(\frac{E}{\alpha\rho_1}\right)^{2/5} t^{-6/5}\mu^{6/5}(4-3\mu)^{-5}\left(\frac{5}{2}-\frac{3}{2}\mu\right)^{82/15} \end{aligned} \tag{4.1}$$

We take the functions v, ρ and p determined by these equations to be functions denoted by primes in the relations (1.12). Then, replacing r and t in equations (4.1) by

$$\frac{r}{\lambda t - \lambda t_0 + 1}, \qquad \frac{t - t_0}{\lambda(t - t_0) + 1} = \tau \tag{4.2}$$

respectively, we obtain an exact solution in parametric form of the problem of a strong point explosion in a uniformly expanding and compressing space

$$\begin{aligned} r &= \left(\frac{E}{\alpha\rho_1}\right)^{1/5}\tau^{2/5}(\lambda t - \lambda t_0 + 1)\,\mu^{-2/5}(5\mu - 4)^{2/13}\left(\frac{5}{2}-\frac{3}{2}\mu\right)^{-82/195} \\ v &= \frac{3}{10}\left(\frac{E}{\alpha\rho_1}\right)^{1/5}\tau^{-3/5}\frac{\mu^{3/5}(5\mu-4)^{2/13}}{\lambda t - \lambda t_0 + 1}\left(\frac{5}{2}-\frac{3}{2}\mu\right)^{-82/195} + \frac{\lambda r}{\lambda(t - t_0) + 1} \\ \rho &= 4\rho_1\,(\lambda t - \lambda t_0 + 1)^{-3}(5\mu - 4)^{9/13}(4 - 3\mu)^{-6}\left(\frac{5}{2}-\frac{3}{2}\mu\right)^{82/13} \\ p &= \frac{3}{25}\rho_1\left(\frac{E}{\alpha\rho_1}\right)^{2/5}\tau^{-6/5}(\lambda t - \lambda t_0 + 1)^{-5}\mu^{6/5}(4-3\mu)^{-5}\left(\frac{5}{2}-\frac{3}{2}\mu\right)^{82/15} \end{aligned} \tag{4.3}$$

To obtain the law of motion of the shock wave and the values of the quantities immediately behind it, it is necessary to assume that $\mu = 1$ in the last equations. Then

$$r_2 = \left(\frac{E}{\alpha\rho_1}\right)^{1/5}\left[\frac{t - t_0}{\lambda(t - t_0) + 1}\right]^{2/5}(\lambda t - \lambda t_0 + 1) \tag{4.4}$$

$$\begin{aligned} v_+ &= \frac{3}{10}\left(\frac{E}{\alpha\rho_1}\right)^{1/5}\left[\frac{t - t_0}{\lambda(t - t_0) + 1}\right]^{-3/5}(\lambda t - \lambda t_0 + 1)^{-1} + \\ &\quad + \left(\frac{E}{\alpha\rho_1}\right)^{1/5}\lambda\left[\frac{t - t_0}{\lambda(t - t_0) + 1}\right]^{2/5} \\ p_+ &= \frac{3}{25}\rho_1\left(\frac{E}{\alpha\rho_1}\right)^{2/5}\left[\frac{t - t_0}{\lambda(t - t_0) + 1}\right]^{-6/5}(\lambda t - \lambda t_0 + 1)^{-5} \end{aligned} \tag{4.5}$$

$$\rho_+ = \frac{4\rho_1}{(\lambda t - \lambda t_0 + 1)^3} \tag{4.6}$$

Outside the shock wave, $p \equiv 0$, and v and ρ are determined by equation (3.11). The expression for the mass of gas inside the shock wave has the form

$$M = \rho_1(\lambda t - \lambda t_0 + 1)^{-3}\,\frac{4}{3}\pi r_2^3 = \frac{4}{3}\pi\rho_1\left(\frac{E}{\alpha\rho_1}\right)^{3/5}\left[\frac{t-t_0}{\lambda(t-t_0)+1}\right]^{6/5} \qquad (4.7)$$

For $\lambda > 0$ (expansion) at $t \to \infty$, and for $\lambda < 0$ (compression), we have for $t \to t^\circ = t_0 - 1/\lambda$, respectively

$$\lim_{t\to\infty} M = M_\infty = \frac{4}{3}\pi\rho_1\left(\frac{E}{\alpha\rho_1}\right)^{3/5}\lambda^{-6/5} \quad (\lambda > 0), \qquad \lim_{t\to t^\circ} M = \infty \quad (\lambda < 0) \qquad (4.8)$$

In the case $\lambda > 0$ the shock wave will "overtake" only those particles in its motion which at $t = t_0$ were contained in the sphere, the radius of which is determined by the expression

$$r = \left(\frac{E}{\alpha\rho_1}\right)^{1/5}\lambda^{-2/5} \qquad (4.9)$$

For $\lambda > 0$ the shock wave will "overtake" in its motion (at $t \to t_0 - 1/\lambda$) every gas particle. For $\lambda < 0$ the expression (4.5) for r_2 vanishes at $t = t_0$ and $t = t_0 - 1/\lambda$.

The maximum value of r_2 is equal to

$$r_2 = r_{2\max} = \frac{3}{5}\left(-\frac{2}{3\lambda}\right)^{2/5}\left(\frac{E}{\alpha\rho_1}\right)^{1/5} \qquad \left(\text{for } t - t_0 = -\frac{2}{5}\frac{1}{\lambda}\right) \qquad (4.10)$$

Thus, for $\lambda < 0$ the law of variation of r_2 with time for $p_1 = 0$ always has a non-monotonic character. Expression (4.5) is easily transformed into the form

$$v_+ = \left(\frac{E}{\alpha\rho_1}\right)^{1/5}(t-t_0)^{-3/5}\,[\lambda(t-t_0)+1]^{-2/5}\left[\frac{3}{10} + \lambda(t-t_0)\right] \qquad (4.11)$$

For $\lambda < 0$ it changes its sign at $t = t_0 - 3/10\,\lambda^{-1}$, i.e. before the radius of the wave begins to decrease. For $t \to t_0 - 1/\lambda$ we have $v_+ \to -\infty$, i.e. the velocity behind the shock wave is directed toward the center of the explosion. If $\lambda > 0$ and the value of the parameter τ_∞", defined by equation (3.16), is sufficiently small; then formulas (4.3) and (4.2) describe with sufficient exactness the motion in the entire infinite range of time $0 < t - t_0 < \infty$. If $\lambda < 0$, then for $p_1 \neq 0$, as was shown above, the shock wave becomes acoustic for $t \to t_0 - 1/\lambda$, and therefore the given equations are valid only in a certain initial portion of the interval $0 < t - t_0 < -1/\lambda$. We note that the exact solution obtained for a strong explosion for $p_1 = 0$ is no longer a similarity solution; it depends on three (dimensional) parameters: ρ_1, E and λ. The parameter t_0 is not essential, because the quantity $t - t_0$ may be substituted for the quantity t, thereby changing the origin of time.

In conclusion it should be noted that the reviewers of this paper competently pointed out that the invariant transformations (1.4), (1.5) and (1.6) may be derived from the results of Ovsiannikov [4]. This book, unfortunately became known to the author after the paper had been submitted for publication.

BIBLIOGRAPHY

1. Kochin, N.E., Kibel', I.A. and Roze, N.W., *Teoreticheskaia gidrodinamika (Theoretical Hydrodynamics)*. OGIZ, 1948.
2. Sedov, L.I., *Metody podobiia i razmernosti v mekhanike (Methods of similarity and dimensionality in mechanics)*. Gostekhteoretizdat, 1954.
3. Korobeinikov, V.P., Mel'nikova, N.S. and Riazanov, E.V., *Teoriia tochechnogo vzryva (Theory of the point explosion)*. Fizmatgiz, 1961.
4. Ovsiannikov, L.V., *Gruppovyie svoistva differentsial'nykh uravnenii (Group properties of differential equations)*. Izd. so Akad. Nauk SSSR, Novosibirsk, 1962.

Translated by J.R.W.